Ward identities for surface-growth models with diffusion

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(Received 27 December 1993; revised manuscript received 8 June 1994)

Using the method of Hwa and Fisher [Phys. Rev. B **49**, 3136 (1994)], we derive Ward identities relating the two- and three-point vertex functions in the dynamic functional theory for surface-growth models with diffusion. These identities result from the statistical transformation symmetries in the underlying Langevin equations for these models. An explicit scaling form for the full three-point vertex function is also obtained. As a result one can show that the scaling properties of the full three-point vertex function are the same as those of the "bare" vertex function obtained in the case of no stochastic noise. This implies that the coupling constant for the nonlinear term in the Langevin equation is not renormalized.

PACS number(s): 05.70.Ln, 64.60.Ht, 68.35.Rh, 05.40.+j

I. INTRODUCTION

Recently Hwa and Fisher [1] applied dynamic functional theory [2,3] to study directed polymers in random media. They analyzed the large-scale fluctuations in the low-temperature pinned phase of a directed polymer using the noisy Burger equation [4]. Because of the invariance of the noisy Burger equation under the "Galilean transformation," they were able to derive useful Ward identities relating different vertex functions in the dynamic functional theory. Besides describing directed polymers in random media, the noisy Burger equation can also be transformed into the Kardar-Parisi-Zhang (KPZ) equation which describes surface-growth models in which evaporation of adatoms and sideway growth are important [5,6].

In the important case of molecular beam epitaxial growth, however, evaporation of adatoms is negligible but diffusion of adatoms on the surface is important. In such a case, various Langevin equations analogous to the KPZ equation had been proposed to describe the large-scale properties of surface growth with diffusion [7,8]. In principle, the dynamic functional approach can also be applied to study these equations, although so far this has not yet been done. This would be a systematic renormalization of the dynamic field theory defined by the dynamic functional derived from these Langevin equations. In such a case it would be useful to derive Ward identities resulting from the transformation symmetries of the underlying Langevin equations, as was done by Hwa and Fisher for the noisy Burger equations. The purpose of this paper is to derive Ward identities for the surface-growth models with diffusion using the method of Hwa and Fisher.

In Sec. II we will present the Langevin equations describing these models. In Sec. III we will describe the mapping of these equations to the dynamic functional which is the starting point of the dynamical field theory for the model. In Sec. IV we will derive the Ward identities resulting from the statistical transformation symmetries of the underlying equations. From these, an explicit scaling form for the full three-point function can be obtained using an assumed scaling form for the linear response function. One can then show that the scaling properties of the full three-point vertex function are the same as those of the bare three-point vertex function obtained in the case of no stochastic noise. Section V provides a summary and conclusions.

II. SURFACE-GROWTH MODELS WITH DIFFUSION

We will be concerned here with the models proposed by Sun, Guo, and Grant (SGG) [7] and Lai and Das Sarma (LDS) [8]. Both models describe the large distance and long time scale behaviors of surface-growth models with diffusion of adatoms, using Langevin equations of the form

$$\frac{\partial h}{\partial t} = -\nu \nabla^2 \left[\nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 \right] + \eta(\mathbf{x},t) \; . \tag{1}$$

Here $h(\mathbf{x},t)$ is the height of the surface at the position $\mathbf{x}$ of a d-dimensional substrate at time t and λ and ν are parameters. $\eta(\mathbf{x},t)$ is a stochastic noise whose correlation is different for SGG and LDS. For SGG, the noise is conservative with correlations

$$\begin{aligned} &\langle \eta \rangle = 0 \; , \\ &\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t') \rangle = -2D\nabla^2 \delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t') \; , \end{aligned} \tag{2}$$

where D is a parameter characterizing the strength of the stochastic noise. For the LDS model, the noise is nonconservative, with correlations

$$\begin{aligned} &\langle \eta \rangle = 0 \; , \\ &\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t) \rangle = -2D\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t') \; . \end{aligned} \tag{3}$$

It turns out that the Ward identities are the same for both SGG and LDS models because the symmetry transformation is the same in both cases. Therefore we need to describe only one of the models, the SGG model.

III. MAPPING TO STOCHASTIC DYNAMICS

The dynamic functional [2,3] for Eq. (1) has the form

$$S[\hat{h},h]=\int dt\int d^dx\left\{-D\nabla^2\hat{h}^2(\mathbf{x},t)+\hat{h}(\mathbf{x},t)\left[\partial_t h(\mathbf{x},t)+\nu\nabla^2\left[\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2\right]\right]\right\}. \quad (4)$$

It is convenient to go to Fourier space where

$$S[\hat{h},h]=\int\frac{d\omega}{2\pi}\int\frac{d^dk}{(2\pi)^d}[Dk^2|\hat{h}(\mathbf{k},\omega)|^2+(\nu k^4-i\omega)\hat{h}(-\mathbf{k},-\omega)h(\mathbf{k},\omega)]$$
$$+\nu\lambda\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}[(k_1^2+k_2^2)\mathbf{k}_1\cdot\mathbf{k}_2+2(\mathbf{k}_1\cdot\mathbf{k}_2)^2]\hat{h}(-\mathbf{k}_1-\mathbf{k}_2,-\omega_1-\omega_2)h(\mathbf{k}_1,\omega_1)h(\mathbf{k}_2,\omega_2)\,. \quad (5)$$

The dynamic functional $S[\hat{h},h]$ specifies completely the field theory for the model in stochastic dynamics. In the absence of stochastic noise, i.e., for $D=0$, this is just the "bare" generating functional $\Gamma^{(0)}[\hat{h},h]$ of the vertex functions.

From (5) the bare vertex functions are easily obtained as

$$\hat{\Gamma}^{(0)}_{1,1}(\mathbf{k},\omega)=\nu\mathbf{k}^4-\mathrm{i}\omega\,, \quad (6)$$

$$\hat{\Gamma}^{(0)}_{1,2}(\mathbf{k}_1,\omega_1;\mathbf{k}_2,\omega_2)=\nu\lambda[(k_1^2+k_2^2)\mathbf{k}_1\cdot\mathbf{k}_2+2(\mathbf{k}_1\cdot\mathbf{k}_2)^2]\,. \quad (7)$$

Because of the relevance of the stochastic noise term in surface-growth dynamics, the renormalized functional $\Gamma[\hat{h},h]$ cannot be obtained exactly. As Hwa and Fisher did, we will establish that the *scaling* behavior obtained from $\hat{\Gamma}^{(0)}_{1,2}$ and $\hat{\Gamma}_{1,2}$ are the same, due to a statistical symmetry transformation on (1). This can yield important information on the properties of the Langevin equation, such as the linear and nonlinear responses of the height h to small perturbations. This is because the three-point vertex function $\hat{\Gamma}_{1,2}$ is related to the linear and nonlinear response functions $\partial h/\partial\hat{J}=G$ and $\delta^2H/\delta^2\hat{J}=G_{2,1}$ of (1), where $\hat{J}$ is a small "perturbation" added to the right hand side of (1).

IV. STATISTICAL TRANSFORMATION INVARIANCE AND WARD IDENTITIES

The Langevin equation (1) is invariant under the transformations [7]

$$h\to h+\mathbf{a}\cdot\mathbf{x}\,, \quad (8a)$$

$$\mathbf{x}\to\mathbf{x}-\nu\lambda t\mathbf{a}\nabla^2\,. \quad (8b)$$

The Fourier transform of (8a) becomes

$$h'(\mathbf{q},t)=\int h(\mathbf{x}-\nu\lambda\mathbf{a}t\nabla^2,t)e^{i\mathbf{q}\cdot\mathbf{x}}d^dx-i\mathbf{a}\cdot\nabla_{\mathbf{q}}\delta^d(\mathbf{q})\,. \quad (9)$$

Differentiating (9) with respect to $\mathbf{a}$ and setting $\mathbf{a}$ to zero, we have

$$\left.\frac{\partial h'}{\partial\mathbf{a}}\right|_{\mathbf{a}=0}=\lambda\nu q^2t\int(\nabla h)e^{i\mathbf{q}\cdot\mathbf{x}}d^dx-i\nabla_{\mathbf{q}}\delta^d(\mathbf{q})\,. \quad (10)$$

Performing an integration by parts on the first term, we have

$$\left.\frac{\partial h'}{\partial\mathbf{a}}\right|_{\mathbf{a}=0}=-i\lambda\nu q^2\mathbf{q}th(\mathbf{q},t)-i\nabla_{\mathbf{q}}\delta^d(\mathbf{q})\,. \quad (11)$$

Since (1) is invariant under the transformation (8), by following Appendix C of Ref. [1] and using (11), we obtain the following identity in Fourier space:

$$\nabla_{\mathbf{q}}\int dt_1\hat{\Gamma}_{1,2}(\mathbf{q},t-t_1;\mathbf{k}-\mathbf{q},t-t_2)|_{\mathbf{q}=0}$$
$$=-\lambda\nu(t-t_2)k^2\mathbf{k}\hat{\Gamma}_{1,1}(\mathbf{k},t-t_2)\,. \quad (12)$$

Transforming to frequency space (12) takes the form

$$\nabla_{\mathbf{q}}\hat{\Gamma}_{1,2}(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\omega-\mu)|_{\mathbf{q},\mu\to 0}=i\lambda\nu k^2\mathbf{k}\frac{\partial}{\partial\omega}\hat{\Gamma}_{1,1}(\mathbf{k},\omega)\,. \quad (13)$$

This is the main result of the paper. It is a relation between the three- and the two-point vertex functions. Similar relations can be obtained for higher order vertex functions, with vertices of n and $(n+1)$ order entering each identify. Comparing with Eq. (C15) of Ref. [1], we can see that this Ward identity differs from the case of KPZ only by the extra factor of k^2 on the right hand side.

In the case of no stochastic noise, we have

$$\hat{\Gamma}^{(0)}(\mathbf{k},\omega)=\nu k^4-i\omega\,. \quad (14)$$

The Ward identity then reads

$$\nabla_{\mathbf{q}}\hat{\Gamma}^{(0)}_{1,2}(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\omega-\mu)|_{\mathbf{q},\mu\to 0}=\lambda\nu k^2\mathbf{k}\,. \quad (15)$$

Since $\hat{\Gamma}_{1,2}$ must be symmetric in $\mathbf{k}_1$ and $\mathbf{k}_2$, this yields exactly the same "bare" three-point vertex as given in (7), which had been obtained directly from the dynamic functional $S[\hat{h},h]$. The two-point vertex function $\hat{\Gamma}_{1,1}$ is the inverse of the linear response function G defined before [9].

As in the case of directed polymer [1] or the KPZ equation [10], the scaling form for the linear response function $\hat{G}(\mathbf{k},t)$ is assumed to be

$$\hat{G}(\mathbf{k},t)\cong\hat{g}(kt^{\zeta})\,, \quad (16)$$

with an exponent ζ.

Using this scaling form for the linear response function and following Appendix C of Ref. [1], the Ward identity (15) now leads to the relation

$$\hat{\Gamma}_{1,2}(\mathbf{q},\mu;\mathbf{k}-\mathbf{q},\omega-\mu)|_{\mathbf{q},\mu\to 0}$$
$$=\lambda\nu k^2\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})\gamma(\omega/k^{1/\zeta})\,, \quad (17)$$

where $\gamma(s)$ is another dimensionless (but complex) scaling function. The above limiting behavior of $\hat{\Gamma}_{1,2}$ suggests a natural scaling form for the full spatial and temporal dependence of the three-point vertex function:

$$\hat{\Gamma}_{1,2}(\mathbf{k}_1,\omega_1;\mathbf{k}_2,\omega_2) = \lambda\nu[(k_1^2+k_2^2)(\mathbf{k}_1\cdot\mathbf{k}_2)+2(\mathbf{k}_1\cdot\mathbf{k}_2)^2] \times V\left[\omega_2\Big/k_2^{1/\zeta},\frac{k_1}{k_2},\frac{\omega_1}{\omega_2}\right], \tag{18}$$

where V is another complex, dimensionless scaling function with $V(s,0,0)=\gamma(s)$. The scaling properties derived from the full three-point vertex and those obtained from the bare vertex function (7) are identical. A similar result was reported by Hwa and Fisher for the KPZ equation. It follows from the scaling form (18) and mild convergence conditions on the function V. This implies that the parameter λ in the nonlinear term in the Langevin equation (1) is not renormalized, as was found in Refs. [7,8].

V. SUMMARY AND CONCLUSIONS

We have derived a Ward identify relating the two- and three-point vertex functions in the dynamic functional theory for surface-growth models with diffusion of the adatoms. These models describe the large spatial and temporal behaviors of molecular beam epitaxial growth in which desorption of adatoms is negligible. This Ward identity results from a statistical transformation symmetry in the underlying Langevin equations of the models. An explicit scaling form for the full three-point vertex function is also obtained. As a result, one can show that the scaling properties of the full three-point vertex function are the same as those of the "bare" vertex function obtained in the case of no stochastic noise. This has the consequence that the coupling constant for the nonlinear term in the Langevin equation is not renormalized. This Ward identity would also be useful in a full renormalization of the dynamic field theory.

ACKNOWLEDGMENTS

The authors thank Dr. T. Hwa and Dr. D. S. Fisher for sharing their paper before its publication and Ms. Kema L. Williams and Mr. Carl R. Grant for valuable typesetting assistance. This work was supported, in part, through the Timbuktu Academy, by the National Science Foundation (Grant No. HRd-9108590), and by the Department of the Navy, Office of Naval Research (ONR Grant No. N00014-93-1-1368).

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